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DEFORMATION OF A STOCHASTICALLY INHOMOGENEOUS SOLID WITH AN OPENING

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At the present time, there exist a very large number of solutions of the deformation of unbounded stochastically inhomogeneous bodies. In these solutions effective moduli of elasticity are determined, i.e., the mechanical properties of the material, averaged over the spatial region, as a function of the parameters characterizing the structural inhomogeneity of the medium. However, by virtue of the nonlocal character of the connection between the mean stresses and the mean deformations [1], the effective moduli of elasticity depend to a considerable degree on the boundary-value problem. It must be noted that, for the solution of concrete boundary-value problems in the theory of elasticity, considerable mathematical difficulties arise. At the present time, there exist solutions to a number of boundary-value problems of the stochastically inhomogeneous theory of elasticity for a half-plane, a band [1], and an infinite plane with a circular opening [2, 3]. In the case of antiplane deformation, a solution has been given to the problem of the propagation of a crack in a stochastically inhomogeneous body [4, 5].

We consider the plane problem of the deformation of a body, whose elastic constants are random functions of the coordinates. With the surface forces g_i given at the contour L of the region S, occupied by the body, and in the absence of volumetric forces, the equations of the plane problem of the theory of the elasticity of isotropic inhomogeneous bodies, written in terms of the stresses, have the form [1]

$$\begin{gathered} X_n = \sigma_x \cos nx + \tau_{xy} \cos ny = g_1, \\ y_n = \tau_{xy} \cos nx + \sigma_y \cos ny = g_2, \\ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \ \nabla^2 [\gamma (\sigma_x - \sigma_y)] = \frac{\partial^2 q}{\partial x^2} \sigma_x + 2 \frac{\partial^2 q}{\partial x \partial y} \tau_{xy} + \frac{\partial^2 q}{\partial y^2} \sigma_y, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \end{gathered} \tag{1}$$

Here q and γ are expressed in terms of the Young modulus E(x, y) and the Poisson coefficient ν(x, y) by the relationships

$$\gamma = 1/E, \ q = (1 + \nu)/E, \tag{2}$$

where ∇^2 is a Laplace operator.

Let q(x, y) and γ(x, y) be random functions of the coordinates. Then relationships (1) constitute a stochastically nonlinear problem, determining the random functions τ_{ij}(x, y). We represent the values of q and γ in the form $q = \langle q \rangle + q'$, $\gamma = \langle \gamma \rangle + \gamma'$. We postulate that the random functions q(x, y) and γ(x, y) are statistically homogeneous and are statistically homogeneously interconnected ($\langle q \rangle$ = const, $\langle \gamma \rangle$ = const). If the solution of the problem (2) is represented in the form of a series in powers of some parameter $\varkappa$, then the problem (1) is normalized. The parameter $\varkappa$ is introduced by the relationships [1].

$$q = \langle q \rangle + \varkappa q', \ \gamma = \langle \gamma \rangle + \varkappa \gamma', \ \tau_{ij} = \sum_{k=0}^{\infty} \varkappa^k \tau_{ij}^{(k)}. \tag{3}$$

Substituting (3) into (1), and equating expressions with identical powers of $\varkappa$, we obtain a boundary-value problem for the zero approximation,

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$$\frac{\partial}{\partial z}(\sigma_x^0 - \sigma_y^0 + 2i\tau_{xy}^0) + \frac{\partial}{\partial \bar{z}}(\sigma_x^0 + \sigma_y^0) = 0, \ \langle\gamma\rangle \nabla^2(\sigma_x^0 + \sigma_y^0) = 0,$$

$$(X_n^0 + iY_n^0)|_L = g_1 + ig_2 \quad (z = x + iy),$$

and the recurrent sequence of the statistically linear boundary-value problems for $\tau_{ij}^{(k)}$,

$$\frac{\partial}{\partial z}(\sigma_x^{(k)} - \sigma_y^{(k)} + 2i\tau_{xy}^{(k)}) + \frac{\partial}{\partial \bar{z}}(\sigma_x^{(k)} + \sigma_y^{(k)}) = 0,$$
$$\nabla^2(\sigma_x^{(k)} + \sigma_y^{(k)}) = f^{(k-1)}, \ (X_n^{(k)} + iY_n^{(k)})|_L = 0, \tag{4}$$

where $f^{(k-1)}$ is a certain linear combination of the form

$$\frac{1}{\langle\gamma\rangle}(\sigma_x^{(k-1)} - \sigma_y^{(k-1)} + 2i\tau_{xy}^{(k-1)})\frac{\partial^2 q'}{\partial z^2}.$$

The problem (4) is the problem of the deformation of a homogeneous body with some manner of distribution of the mass forces and a boundary free of external loads.

The solution of this problem is given by the formulas [1, 6, 7]

$$\sigma_x^{(k)} + \sigma_y^{(k)} = \frac{2}{\pi}\iint_{\Omega} f^{(k-1)}(\chi,\bar{\chi}) \ln|\chi - z| \, d\chi_1 d\chi_2 + 2[\Phi_k(z) + \overline{\Phi_k(z)}],$$
$$\sigma_x^{(k)} - \sigma_y^{(k)} + 2i\tau_{xy}^{(k)} = \frac{z}{\pi}\iint_{\Omega} \frac{f^{(k-1)}(\chi,\bar{\chi})\, d\chi_1 d\chi_2}{\bar{\chi} - \bar{z}} - 2[\overline{z\Phi_k'(z)} - \overline{\Psi_k(z)}], \tag{5}$$

where Ω is the transverse cross section of the region occupied by the body; χ_1, and χ_2 are integration variables ($\chi = \chi_1 + i\chi_2$, $\bar{\chi} = \chi_1 - i\chi_2$); and $\Phi_k(z)$ and $\Psi_k(z)$ are complex Muskhelishvili potentials, which assure the satisfaction of the boundary conditions. The potentials $\Phi_k(z)$ and $\Psi_k(z)$ are easily found in all cases where the given contour is mapped on a circle by a rational function.

As an example, we consider the case of the elongation of an infinite stochastically inhomogeneous body with a circular opening (Fig. 1), and we calculate the second approximation of the stresses $\langle\sigma_\theta^{(2)}\rangle$, $\langle\sigma_\rho^{(2)}\rangle$, $\langle\tau_{\rho\theta}^{(2)}\rangle$, where $\langle\sigma_\rho^{(k)}\rangle + \langle\sigma_\theta^{(k)}\rangle = \langle\sigma_x^{(k)}\rangle + \langle\sigma_y^{(k)}\rangle$; $\langle\sigma_\theta^{(k)}\rangle - \langle\sigma_\rho^{(k)}\rangle + 2i\langle\tau_{\rho\theta}^{(k)}\rangle = (\langle\sigma_y^{(k)}\rangle - \langle\sigma_x^{(k)}\rangle + 2i\langle\tau_{xy}^{(k)}\rangle)e^{i\theta}$.

We take the correlation functions in the form [8]

$$\langle q'(z_1)q'(z_0)\rangle = \sigma_{qq}^2 \exp[-(z_1 - z_0)(\bar{z}_1 - \bar{z}_0)/\alpha^2],$$
$$\langle q'(z_1)\gamma'(z_0)\rangle = \sigma_{q\gamma}^2 \exp[-(z_1 - z_0)(\bar{z}_1 - \bar{z}_0)/\alpha^2],$$
$$\langle\gamma'(z_1)\gamma'(z_0)\rangle = \sigma_{\gamma\gamma}^2 \exp[-(z_1 - z_0)(\bar{z}_1 - \bar{z}_0)/\alpha^2].$$

Then after certain transformations, the calculation of expressions (5) reduces to calculation of integrals of the form

$$\int_1^\infty dr_0 \int_1^\infty dr_1 \int_{\gamma_0} d\sigma \int_{\gamma_1} \exp[-R(r_0^2 + r_1^2)/\alpha^2] f(r_0, r_1, \sigma_0, \sigma_1)\, d\sigma_1, \tag{6}$$

where γ_0, and γ_1 are contours of a unit circle; σ_0, and σ_1 are points of the contour. The integrals with respect to γ_0, and γ_1 are calculated using the theory of residues, taking account of in which region the function under the integral sign is holomorphic. The integrals of the form

$$\int_1^\infty dr_0 \int_1^\infty \exp[-R^2(r_1^2 + r_0^2)/\alpha^2]\, dr_1,$$

to which expression (6) is reduced, were calculated on a Mir-1 digital computer [9]. The coefficient of the stress concentration k, being the ratio of $\langle\sigma_\theta^2\rangle$ to the value of the load p, applied at infinity, is shown in Fig. 1, where the solid line illustrates the dependence of the coefficient of the stress concentration in a homogeneous body on the dimensionless distance r/R; the dashed lines show the coefficients of the stress concentration in the case of a stochastically inhomogeneous body. The ratio of the characteristic linear dimension of the inhomogeneity to the radius R is taken as $\alpha/R = 0.1$ and 0.3 (curves 1 and 2, respectively).

The calculations were made for the following values of the parameters:

$$\sigma_{qq}^2/\langle\gamma^2\rangle = 0.1, \ \sigma_{\gamma q}/\langle\gamma^2\rangle = 0.01, \ \sigma_{\gamma\gamma}/\langle\gamma^2\rangle = 0.005.$$

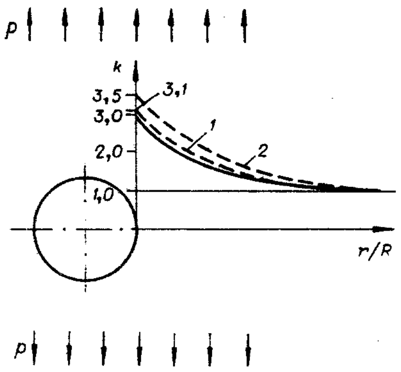

Fig. 1

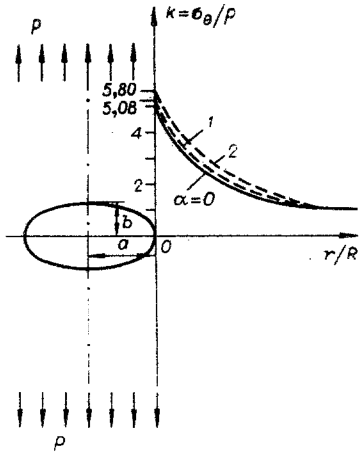

Fig. 2

Thus, the scale effect, characteristic of stochastically inhomogeneous bodies [1], leads to a considerable increase in the stress concentration near the opening.

Calculations were made analogously for the case of an infinite body, weakened by an elliptical opening: The effective coefficient of the stress concentration in the case of an elliptical opening is shown in Fig. 2 for the case $a/b = 2$ [curve 1) $\alpha/R = 0.1$; 2) $\alpha/R = 0.3$].

In the case where the ellipse reverts to a slit, within the framework of the linear theory of elasticity, as is well known, the stresses at the tip of the crack have a singularity of order $(r)^{-1/2}$. In [10] it is shown that the singularity in the neighborhood of the tip of a crack in a stochastically inhomogeneous body is of the same order as in a homogeneous body, i.e., the effective coefficient of the intensity of the stresses can be introduced.

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